

Buffer occupancy asymptotics in rate proportional sharing networks with heterogeneous long-tailed inputs

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November 12, 2014

Abstract

In this paper, we consider a network of rate proportional processor sharing servers in which sessions with long-tailed duration arrive as Poisson processes. In particular, we assume that a session of type n transmits at a rate r_n bits per unit time and lasts for a random time τ_n with a generalized Pareto distribution given by $\mathbb{P}\{\tau_n > x\} \sim \alpha_n x^{-(1+\beta_n)}$ for large x , where $\alpha_n, \beta_n > 0$. The weights are taken to be the rates of the flows. The network is assumed to be loop-free with respect to source-destination routes. We characterize the order O -asymptotics of the complementary buffer occupancy distribution at each node in terms of the input characteristics of the sessions. In particular, we show that the distributions obey a power law whose exponent can be calculated via solving a fixed point and deterministic knapsack problem. The paper concludes with some canonical examples.

Keywords: Queueing networks; Processor sharing; Buffer asymptotics; Pareto distribution; Long tailed distributions.

1 Introduction

Modern networks carry many different types of traffic streams with widely differing characteristics. The statistical variations that are of primary interest are those which are short lived and those that are long flows. The reason is that these two types of flows have very different impacts in terms of engineering the network in terms of router speeds and buffering requirements. The key issue is that long duration flows must be handled carefully and their statistical characteristics usually are the dominant ones in terms of network queueing performance. On the time-scale of long flows, short flows can be viewed as averaged out and while they can take up bandwidth it is predictable and thus the primary situation of interest is one in which long-lived flows are present. A detailed statistical analysis of such sessions suggests that they are well modelled by ON-OFF type of processes where the ON periods, of random duration, have a tail distribution which decays according to a long-tailed distribution. Different definitions are used in the literature for long and heavy tails. Here we assume that a r.v. X is long-tailed if $\mathbb{P}\{X > x\} \sim \text{const. } x^{-\beta}$ with $\beta > 0$. If $\beta \in (0, 2]$, then X has infinite variance and is said to heavy-tailed. These distributions are said to be sub-exponential. When many independent such sessions arrive randomly, the aggregated input process is long-range dependent.

The performance of networks and their ability to offer Quality of Service (QoS) depend on accurately capturing the parametric dependence of the QoS measures such as the delay or loss distributions. While calculating these distributions exactly is intractable, the asymptotics of the tail distribution are much more manageable.

Numerous studies have shown that the presence of long and heavy-tailed traffic forces us to change the way the buffer dimensioning is done in that much more buffering is necessary to achieve similar buffer overflow characteristics than with conventional or short-range dependent traffic. Indeed, the

way buffer overflows occur can be very different. This is essentially due to the way that large excursions of the buffer workload take place. For conventional traffic models, the tail of the stationary workload distribution is exponential while for long-range dependent traffic, it has an asymptotic long-tail or sub-exponential decay.

There are many results available for buffer asymptotics of a single node or server with long-tailed and sub-exponential inputs. These have been obtained under different hypotheses. These hypotheses relate to conditions on the session length distribution decay rates, session transmission rates and session arrival models. All results deal with stationary queues where the average rate is less than capacity. The vast majority of results are for FIFO systems where the source transmission rates are identical, which is referred as the homogeneous case. These can be found, for example, in [11, 19, 22, 17, 26, 27]. An excellent survey for the homogeneous case can be found in [8]. It has however been shown in [21] that assuming the same transmission rates can lead to erroneous conclusions on the asymptotic behavior of the tail distribution of the buffer length when the inputs have long-tailed session lengths. This was obtained for the so-called $M/G/\infty$ model where sessions arrive according to a Poisson process, transmit at different rates and have differing long-tailed random session holding times. Similar results and conclusions have also been obtained for a fixed number of ON-OFF sources in [7]. Since these early papers there have been a number of works on studying the effects of sub-exponential distributions and how they arise in the context of network traffic, see [19] and references therein.

The extension of above results to a general network is quite difficult since the traffic loses its simple parametric structure after passing through its entrance node. Networks with exponentially distributed service times were considered in [3] and [14]. In [23], the author studied the large deviations problem for feedforward networks in heavy traffic. The papers were basically concerned with the computation of the rate functions associated with the tail distributions of the buffer occupancy. This is a difficult variational problem. In general, when the input and output rate functions in queues have the so-called “linear geodesics” an end-to-end analysis is feasible by an iterative procedure. In [15, 24], it was shown that in queues with many inputs the linear geodesic property does not hold and thus calculating the rate functions of the outputs is difficult.

There are very few results available for networks with long-tailed service times. In [6], the authors consider the (max, plus) setting and derive asymptotics of the distributions of total response times for networks which include tandem queues. In [18], a feedforward network is analyzed when a node with sub-exponential service time has upstream nodes with lighter-tailed service times. In spite of these early successes the network case with general long-tailed have been analyzed only for very special cases such as feed-forward, or tandem networks [20]. Most network models assume service times are i.i.d at nodes with sub-exponential distributions. In reality, flows do not change their sizes and thus the analysis must take into account the fact that within the networks the service times are no more independent. A recent issue [1] devoted to network asymptotics does not address the issue of network buffer distributions. Thus, except for special network models that do not correspond to flow type session traffic there are no explicit results available that characterize the stationary buffer distributions.

In this paper we consider the model where sessions arrive independently as Poisson processes and transmit at a fixed rate during their session duration. At a given node when the server capacity is exceeded, the capacity is shared in proportion to the rates of the flows else the source passes through unchanged. This is often referred to as rate proportional sharing and can be viewed as a discriminatory processor sharing discipline (DPS) [2] that approximates a weighted based round-robin service where the weights correspond to the rates of the flows entering the router. The rate multiplied by the number of flows denotes the aggregated required bandwidth of the flows. The session or flow durations are assumed to be long-tailed, having an asymptotic power law or generalized Pareto decay. The sessions are assumed to be heterogeneous so that they have different session duration distributions and different

transmission rates. This gives rise to an heterogeneous $M/G/\infty$ type of model for the inputs. It is assumed that the types of sessions are defined by the routes that they take through the network and it is assumed that paths are loop-free. Although the path for a single end-to-end route is loop free, the interaction between the flows in the network creates dependence between flows within the network, i.e., the input characterization is changed. In particular the probabilistic characterization of the duration and rates changes due to the sharing of the processors. We obtain the O -asymptotics (i.e. we identify the asymptotic power law decay) for the buffer occupancies at each node. In particular it shows that the characterization of the decay rate as the solution an optimal knapsack problem for a single server carries through to the network case with changes in the rates. The utility of this result is that we can identify the sources that are the most problematic and also obtain rough estimates for the loss and delay distributions. These results extend to networks with rate proportional sharing the results that were obtained in [21] however it is now more complicated due to the fact that output processes are difficult to characterize in non-Markovian settings. In the single node case, exact asymptotics of buffer occupancy and loss obtained in [21] showed that these asymptotics are not only governed by the tail distribution of sources but also depend on their transmission rates that could be obtained from a knapsack problem. This result was also obtained in [28] using the idea of equilibrium service times pioneered by Pakes [25] for tails sub-exponential inputs to a queue.

In this paper we show that a similar characterization also holds in a general loop-free network with rate proportional sharing. However computing the exponents of the buffer tail distributions need us to first solve a fixed point problem. This is because the interaction of traffic inside the network causes transmission rates and average load rates to be modified during long buffer exceedance times. Furthermore, since we do not assume that the network is feedforward, it results in a fixed point equation for computing the “modified” rates. It is difficult to improve these results (e.g. obtain exact asymptotics) since the Poisson arrival structure and independence of sessions are lost for the traffic inside the network.

The organization of this paper is as follows: In Section 2, we formulate the model and present the preliminaries. Section 3 contains the main result with the proofs. An example of a two node network is considered in Section 4. In Section 5, we give a discussion of results and concluding remarks.

2 Model and Preliminaries

We consider a discrete-time fluid DPS model where traffic arrivals and services take place in slots indexed by $t \in \mathbb{Z}$ with the convention that arrivals take place at the beginning of a slot and services are completed at the end of the slot. This can be seen as corresponding to a situation in the continuous time where the arriving traffic coming in $(i, i + 1]$ are all served at time $i + 1$. We refer to t as the time instant in the discrete-time model. There is a finite set \mathcal{N} of traffic types (classes) with $N = \text{card}(\mathcal{N})$ which are differentiated according to their transmission rates and their session lengths and different classes are assumed to be mutually independent. Session requests for type $n \in \mathcal{N}$ arrive randomly according to a Poisson process with rate λ_n . Let θ_t^n be the number of sessions of class n arriving at time t . We assume that θ_t^n are i.i.d. for fixed n and

$$\mathbb{P}\{\theta_t^n = k\} = e^{-\lambda_n} \frac{\lambda_n^k}{k!}$$

A session of class n then transmits at the rate r_n (bits/slot) for a duration τ_n (slots) which is assumed to have a long-tailed distribution. Let $\tau_{t,j}^n$ denote the session length of the j 'th session of class n arriving at time t . The r.v.'s $\tau_{t,j}^n$ are assumed to be i.i.d. for fixed n and satisfy

$$\mathbb{P}\{\tau_{t,j}^n \geq z\} = \mathbb{P}\{\tau_n \geq z\} \sim \alpha_n z^{-(1+\beta_n)}$$

where $\alpha_n, \beta_n > 0$ and $A(x) \sim B(x)$ means that $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} = 1$. Also $A(x) \preceq (\succeq) B(x)$ means $\limsup_{x \rightarrow \infty} \frac{A(x)}{B(x)} \leq 1 (\liminf_{x \rightarrow \infty} \frac{A(x)}{B(x)} \geq 1)$, i.e., the inequalities are in an asymptotic sense. This model is a generalization of the $M/G/\infty$ model proposed by Cox [9], which we refer to as the $(M/G/\infty)^N$ model and is depicted in Fig. 1.

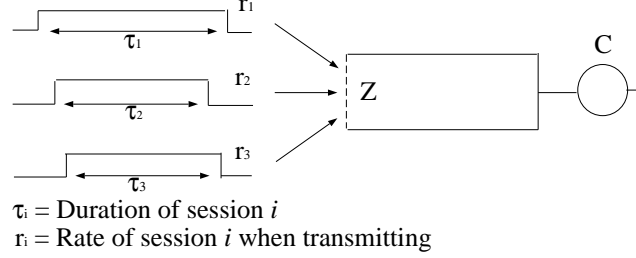


Figure 1: Model of arriving sessions

The network is composed of M nodes (see Fig. 2 below). It is assumed that the packets from the sessions are admitted into an infinite buffer and the buffer is served at a rate of C_m per unit time at node $m = 1, \dots, M$. The capacity is shared in proportion to the number of sessions and their rates. Type $n \in \mathcal{N}$ traffic has a fixed route without any loops and its path is represented by the vector $\pi^n = [\pi_1^n, \dots, \pi_{l_n}^n]$ where $\pi_i^n \in \{1, \dots, M\}$. Hence, type n traffic traverses the nodes by entering the network at node π_1^n and leaving after node $\pi_{l_n}^n$ and $\pi_i^n \neq \pi_j^n$ for $i \neq j$. We define the set of traffic types which pass through node m by $\mathcal{N}_m \doteq \{n \in \mathcal{N} : \pi_i^n = m, 1 \leq i \leq l_n\}$.

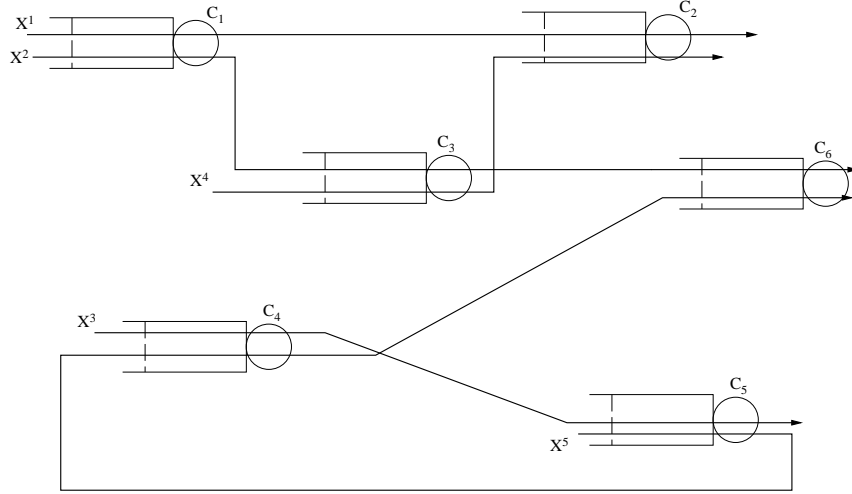


Figure 2: A typical network considered in this paper

Let X_t^n be the input traffic process of class n entering the network. Then,

$$X_t^n = \sum_{i=t}^{-\infty} \sum_{j=1}^{\theta_i^n} r_n \mathbf{1}_{\{\tau_{i,j}^n \geq t-i\}}.$$

and average (mean) load of class n is

$$\rho_n \doteq \lambda_n r_n \mathbb{E}[\tau_n]$$

Let $W_t^{m,n}$ denote the workload due to a flow of type n at server m and

$$W_t^m = \sum_{n \in \mathcal{N}_m} W_t^{m,n}$$

denote the total work due to all flows at server m .

Let

$$\sum_{n \in \mathcal{N}_m} \rho_n < C_m; \quad \forall m \quad (1)$$

Under this hypothesis, the queues in the network are stable since the network is loop-free for all classes of flows. condition, by Loynes' theorem [5], there exists a stationary version of W^m .

Consider the following splitting of the input process X_t^n into two processes $X_t^{n,L}$ and $X_t^{n,H}$ as follows: The process $X_t^{n,L}$ is formed by active sessions which have session lengths at most εz (will be referred as “short” sessions in the context) with $\varepsilon > 0$ and given by

$$X_t^{n,L} = \sum_{i=t}^{t-\varepsilon z} \sum_{j=1}^{\theta_i^n} r_n \mathbf{1}_{\{\varepsilon z \geq \tau_{i,j}^n \geq t-i\}} \quad (2)$$

In the sequel, z will be taken as the large buffer occupancy level and $\varepsilon > 0$ will be chosen to be a small enough number. The process $X_t^{n,H}$ is composed of sessions with lengths greater than εz (will be referred as “long” sessions in the sequel) and given by

$$X_t^{n,H} = \sum_{i=t}^{-\infty} \sum_{j=1}^{\theta_i^n} r_n \mathbf{1}_{\{\tau_{i,j}^n > \varepsilon z, \tau_{i,j}^n > t-i\}} \quad (3)$$

As it can be seen from the definition, the processes $X_t^{n,L}, X_t^{n,H}$ are mutually independent and $X_t^n = X_t^{n,L} + X_t^{n,H}$. The superscript ‘ H ’ indicates that the process is strictly “heavy” (or “long”) and the superscript ‘ L ’ indicates that the process is strictly “light” (or “short”). We will refer to $X^{n,L}$ and $X^{n,H}$ as short and long processes in this context. As $z \rightarrow \infty$, we have $\mathbb{E}X_t^{n,L} \rightarrow \mathbb{E}X_t^n$ and $\mathbb{E}X_t^{n,H} \rightarrow 0$. Despite the fact that $\mathbb{E}X_t^{n,H} \rightarrow 0$ as $z \rightarrow \infty$, the process $X_t^{n,H}$ will contribute significantly to the large buffer occupancy probability.

Later, we will choose ε small enough so that the short sessions can be replaced with constant inputs without any effect on the buffer length O-asymptotics.

We define $X_{k,t}^n$ ($Y_{k,t}^n$) to be the class n input (output) traffic at node k and time t . For any process Z with value Z_t at time t , the cumulative process in the time interval $(t_1, t_2]$ will be denoted by $Z(t_1, t_2)$, i.e., $Z(t_1, t_2) = \sum_{i=t_1+1}^{t_2} Z_i$.

We also need the following definitions to state the main result. Let $J \in \mathbb{Z}_+^N$ where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. J will correspond to the combination of long (longer than εz) sessions; i.e. there are J_n long sessions of class n . If the transmission rates of active long sessions at a time exceed the difference of total output capacity and average load at a node, a scaling effect would occur. This happens because the service capacity is shared among all these active sessions according to rate proportional scheduling and at times when there is not enough output capacity at a node to serve all the traffic without buffering, the output process of this node will behave as a scaled version of its input process. In particular, the mean load of short sessions (which goes to the average workload as $z \rightarrow \infty$) and the transmission rates of long sessions would change at the output. In order to quantify this, we first define $p^n(m) = m'$ for $n \in \mathcal{N}_m$ if $\pi_i^n = m$ and $\pi_{i-1}^n = m'$; i.e. $p^n(m)$ is the upstream node of node m

for class n . Take $p^n(m) = 0$ if $m = \pi_1^n$. We define the transmission rate and mean of class n traffic entering node m as follows:

$$r_m^{n,J} = \begin{cases} 0 & \text{if } n \notin \mathcal{N}_m \\ r_n & \text{if } p^n(m) = 0 \\ r_{p^n(m)}^{n,J} S_{p^n(m)}^J & \text{otherwise} \end{cases} \quad (4)$$

and

$$\rho_m^{n,J} = \begin{cases} 0 & \text{if } n \notin \mathcal{N}_m \\ \rho_n & \text{if } p^n(m) = 0 \\ \rho_{p^n(m)}^{n,J} S_{p^n(m)}^J & \text{otherwise} \end{cases} \quad (5)$$

where

$$S_m^J = \frac{C_m}{\max\{C_m, \sum_{n \in \mathcal{N}_m} J_n r_m^{n,J} + \rho_m^{n,J}\}} \quad (6)$$

S_m^J is just the fraction of the capacity available to each flow at the node m and so $r_n S_m^J$ would be the rate allocated to flow n at node m with configuration of J long flows. Note that the output traffic is no longer $M/G/\infty$.

The function S_m^J measures the scaling of sources for the output transmission at node m for a combination of J active long sessions. Note that $0 < S_m^J < 1$ when the input rates of the long sessions exceed the difference between the output capacity and the total average loads at node m and $S_m^J = 1$ if this does not happen.

Lemma 2.1. *For every $J \in \mathbb{Z}_+^N$, equations (4) and (5) are well defined, i.e., there exist unique solutions $r_m^{n,J}$ and $\rho_m^{n,J}$ satisfying these equations. Furthermore, for fixed J , $r_m^{n,J}$ and $\rho_m^{n,J}$ are continuous in parameters r_n and ρ_n .*

Proof. See Appendix A.

We now define the following for future usage:

$$R^J = \sum_{n \in \mathcal{N}} J_n r_n, \quad R_m^J = \sum_{n \in \mathcal{N}_m} J_n r_m^{n,J} \quad (7)$$

$$\rho = \sum_{n \in \mathcal{N}} \rho_n, \quad \rho_m^J = \sum_{n \in \mathcal{N}_m} \rho_m^{n,J} \quad (8)$$

$$\kappa_J = \sum_{n \in \mathcal{N}} J_n \beta_n, \quad J_0^m = \arg \min_J \{\kappa_J : R_m^J > C_m - \rho_m^{n,J}\} \quad (9)$$

$$\beta_{\min} = \min_{n \in \mathcal{N}} \beta_n, \quad r_{\max} = \max_{n \in \mathcal{N}} r_n \quad (10)$$

The results below, taken from [21] (Lemma 3.3 and Lemma 3.4 therein), provide bounds for the deviations of a short process from its average rate.

Lemma 2.2. *For any $n \in \mathcal{N}$ and sufficiently small $\delta_1 > 0$, $\delta_2 > 0$ and sufficiently large z*

$$\mathbb{P}\{\inf_{t \geq 0} \{X^{n,L}(-t, 0) - (\rho_n - \delta_1)t\} < -\delta_2 z\} \leq e^{-O(z)}.$$

Lemma 2.3. *For any $n \in \mathcal{N}$ and $\delta_2^0 > 0$, $\delta_1 > 0$, $c_1 > 0$ and*

$0 < \varepsilon < \min\{\frac{\delta_2^0}{2(\rho_m + \delta_1)}, \min_i \frac{\beta_i^}{2c_1 r_i}\}$ where $\beta_i^* = \beta_i$, $0 < \beta_i < 1$ and $\beta_i^* = \frac{1}{2}$, $\beta \geq 1$, if z is sufficiently large, then uniformly over all $\delta_2 > \delta_2^0$*

$$\mathbb{P}\{\sup_{t \geq 0} \{X^{n,L}(-t, 0) > t(\rho_n + \delta_1)\} > \delta_2 z\} \leq z^{1-c_1 \bar{\delta}_2}$$

where $\bar{\delta}_2 \doteq \delta_2 - (\rho_m + \delta_1)\varepsilon > 0$.

3 Main Result and Proofs

Our goal is to obtain the O -asymptotics for the probability that the buffer content at node m exceeds a large value z . We will use $f(z) = O(g(z))$ to mean $0 < \liminf_{z \rightarrow \infty} f(z)/g(z) \leq \limsup_{z \rightarrow \infty} f(z)/g(z) < \infty$ and $f(z) = o(g(z))$ to mean $0 < \lim_{z \rightarrow \infty} f(z)/g(z) = 0$. It will be shown that the buffer of node m will reach a large value z if there are enough number of long sessions which contribute to this. During this time, the short sessions show average behavior, i.e., produce traffic around their mean load level. Thus, the short processes can be replaced with constant (equal to the mean load) processes in the asymptotic regime without causing a significant change at large occupancy levels of buffers.

We first review the single node result ($M = 1$) from [21] (Theorem 2.1 therein) where exact buffer asymptotics were obtained. A form of their result, sufficient for our purposes in this paper, is

$$\mathbb{P}\{W_0^1 > z\} = \mathbb{P}\{\sup_{t>0} \sum_{n \in \mathcal{N}_1} X^n(0, t) - C_1 t > z\} = O(z^{-\kappa_{J_0^1}}) \quad (11)$$

It was shown in [21] also that the most likely large buffer occupancy is generated by a busy period when exactly J_0^1 long sessions are active during this busy period. If there is only a single type of flow ($\text{card}(\mathcal{N}) = 1$), then $J_0^1 = \lceil (C_1 - \rho_1)/r_1 \rceil$. Hence the buffer length decay rate $\kappa_{J_0^1}$ depends not only on the decay rates β_n 's but also on the transmission rates r_n 's unless $r_1 > C_1 - \rho_1$ in which case $J_0^1 = 1$.

The basic idea of the proof is the following (that is valid at a single buffer with independent short and long processes):

Let $\{Y_t\}$ denote the input process to a buffer that operates at rate C_m and Y^L and Y^H refer to the short and long inputs defined in (2) and (3) and thus:

$$Y_t = Y_t^L(\varepsilon) + Y_t^H(\varepsilon) \quad (12)$$

with $\mathbb{E}[Y_t] = \sum_{n \in \mathcal{N}_m} \rho_n < C_m$. In the following sketch of the ideas for simplicity we set $\sum_{n \in \mathcal{N}_m} \rho_n = \rho$.

Having defined this decomposition of $\{Y_t\}$ the basic idea of the proof is as follows.

Upper bound

For the upper bound, we choose ε such that $\mathbb{E}[Y_t^H(\varepsilon)] < C_m - \delta_1 - \rho$ for some $\delta_1 > 0$ small. Such a choice always exists since $\mathbb{E}[Y_t] = \rho < C_m$.

Now consider $\{Y_t^L(\varepsilon)\}$ as an input to a queue with service rate $\rho + \delta_1$ and $\{Y_t^H(\varepsilon)\}$ as an input to another queue with service rate $C_m - \rho - \delta_1$. Then both queues are stable and let $\{W_t^{U,L}\}$ denote the stationary workload for the first queue and $\{W_t^{U,H}\}$ denote the stationary workload for the second queue.

Define $X(-t, 0) = \sum_{j=-t}^0 Y_j$, and $X^i(-t, 0) = \sum_{j=-t}^0 Y_j^i(\varepsilon)$; $i = L, H$.

Then the stationary workloads at time 0 are given by:

$$W_0 = \sup_{t \geq 0} \{X(-t, 0) - Ct\}$$

$$W_0^{U,L} = \sup_{t \geq 0} \{X^L(-t, 0) - (\rho + \delta_1)t\}$$

and

$$W_0^{U,H} = \sup_{t \geq 0} \{X^H(-t, 0) - (C - \rho - \delta_1)t\}$$

and hence it readily follows that

$$W_0 \leq W_0^{U,L} + W_0^{U,H}$$

Therefore:

$$\mathbb{P}(W_0 > z) \leq \mathbb{P}(W_0^{U,L} + W_0^{U,H} > z) \quad (13)$$

We can show that the complementary distributions are such that: $\mathbb{P}(W_0^{U,L} > z) \sim o\left(\mathbb{P}(W_0^{U,H} > z)\right)$ and $\mathbb{P}(W_0^{U,H} > z)$ is long-tailed. Noting that $W_0^{U,L}$ and $W_0^{U,H}$ are independent we can then use the following result which can be found in Feller [12, p. 271, Vol. 2] which we state in a slightly extended form below:

Proposition 3.1. *Let $F_1(\cdot)$ and $F_2(\cdot)$ be two distribution functions such that as $x \rightarrow \infty$*

$$1 - F_i(x) \sim a_i x^{-\nu_i}$$

*Then the convolution $G = F_1 * F_2$ satisfies, for $x \rightarrow \infty$*

$$1 - G(x) \sim \alpha x^{-\min(\nu_1, \nu_2)}$$

and α corresponds to the α_i associated to the index of the exponent.

Moreover, as $x \rightarrow \infty$

$$1 - G(x) \sim (1 - F_1(x)) + (1 - F_2(x))$$

Applying the above result it readily follows that:

$$\mathbb{P}(W_0 > z) \leq \mathbb{P}(W_0^{U,H} > z)(1 + o(z))$$

One of the key results that was established in [21] was the notion of an *Isolated Typical Busy Period* which is a busy period associated with the arrival of exactly J_0 long sessions where J_0 is defined as the minimum exponent given by:

$\kappa_J = \sum_{i=1}^N \beta_i j_i$ which corresponds to decay exponent associated with the set J .

$R_J = \sum_{i=1}^N r_i j_i$ the rate corresponding to the set J of sessions.

$$J_0 = (j_1^o, \dots, j_N^o) = \operatorname{argmin}_J \{\kappa_J : R_J - (C - \rho) > 0\}$$

The idea of an ITBP is similar to the role of the *Principle of the Single Large Jump* in the context of sums of sub-exponential random variables as presented in the monograph by Foss *et. al* [13].

Lower bound

The proof of the lower bound requires a finer argument. Choose $\delta_1 > 0$ and then $\varepsilon > 0$ such that : $\mathbb{E}[Y_t^L(\varepsilon)] > \rho - \delta_1$ and $\mathbb{E}[Y_t^H(\varepsilon)] < C_m - \rho + \delta_1$. Once again for any $\delta_1 > 0$ there exists $\varepsilon > 0$ such that these conditions can be met since $\mathbb{E}[Y_t] = \rho$. Now consider two queues with inputs $\{Y_t^L(\varepsilon)\}$, $\{Y_t^H(\varepsilon)\}$ and service rates $\rho - \delta_1$ and $C - \rho + \delta_1$ respectively. Note that the queue corresponding to first input is unstable.

Now with similar notation as above:

$$\sup_{t \geq 0} \{X(-t, 0) - Ct\} \geq \sup_{t \geq 0} \{X^H(-t, 0) - (C - \rho + \delta_1)t\} + \inf_{t \geq 0} \{X^L(-t, 0) - (\rho - \delta_1)t\}$$

Denote by $W_0^{L,H}$ the stationary workload given by:

$$W_0^{L,H} = \sup_{t \geq 0} \{X^h(-t, 0) - (C - \rho + \delta_1)t\}$$

and let

$$W_0^{L,L} = \sup_{t \geq 0} \{(\rho - \delta_1)t - X^l(-t, 0)\}$$

Note, $W_0^{L,L}$ can be viewed as a risk process [4] where $X^L(-t, 0)$ corresponds to the cumulative claims in $[-t, 0]$. Under the assumed condition we know that the ruin time is finite w.p.1.

From above:

$$\mathbb{P}(W_0 > z) \geq \mathbb{P}(W_0^{L,H} + W_0^{L,L} > z)$$

Once again since $W_0^{L,L}$ and $W_0^{L,H}$ are independent r.v.'s, by establishing the fact that $\mathbb{P}(W_0^{L,L} > z) \sim o\left(\mathbb{P}(W_0^{L,H} > z)\right)$ by virtue of Proposition 3.1 it follows that as $z \rightarrow \infty$

$$\mathbb{P}(W_0^{L,H} > z)(1 + o(z)) \leq \mathbb{P}(W_0 > z)$$

establishing a way to obtain the lowerbound. By choice of $\delta_1 \rightarrow 0$ we will then establish that the upper and lower bounds are equal within factors of $o(z)$.

Having given the basic idea of the proof we now proceed with the proofs of the principal results. In the network case we will show that a short process of type n as it enters the network can be replaced with a constant process $\rho_n + \delta$ ($\rho_n - \delta$) for small enough $\delta > 0$ if $n \in \mathcal{N}_m$ ($n \notin \mathcal{N}_m$) while considering the asymptotics of node m .

Then we will show that the busy period where the buffer of node m reaches a large z is created by a combination of long sessions (J) which satisfies $R_m^J > C_m - \rho_m^{n,J}$. Note that the scaling defined before Lemma 2.1 will take place at a node if long sessions have a total rate more than the difference of the output capacity and the average input load at this node. This scaling causes a long session to become longer at the output with a smaller transmission rate. Also its transmission will start after all the traffic in the buffer is served. Therefore we will consider a process which is obtained from the long process by shifting a session by the buffer occupancy at its arrival time and lengthening it by an amount proportional to the total traffic arriving during the lifetime of this session. This shifting and scaling will allow us to bound the time when the service of a session finishes at the downstream nodes. It will be shown that the probability that this new process has J active sessions at $t = 0$ is $O(z^{-\kappa_J})$. Furthermore, the case $R_m^J > C_m - \rho_m^J$ will dominate the other combinations in probability when the buffer occupancy at node m is greater than a large z . The optimal (asymptotically most likely) such combination was defined to be as J_0^m with a probability of $O(z^{-\kappa_{J_0^m}})$ and this will give the upper bound.

The lower bound part is relatively easier. We assume that the J_0^m long sessions arrive in $(-kz, k(1 - \alpha)z)$ where $0 < \alpha < 1$ is small and $k > 0$ is later chosen large enough. This event has a probability of $O(z^{-\kappa_{J_0^m}})$. From the upper bound part, the probability that a buffer has a level greater than $\delta_1 z$ at time $(-kz - 1)$ is $z^{-\gamma}$ for some $\gamma > 0$. For small enough δ_1 , the contribution of buffer contents at time $(-kz - 1)$ to the buffer content of node m at $t = 0$ will be less than $\delta_2 z$ for small $\delta_2 > 0$ with high probability. The short sessions are again replaced with constant processes but this time with $\rho_n - \delta$ if $n \in \mathcal{N}_m$ and $\rho_n + \delta$ if $n \notin \mathcal{N}_m$. Then we show that this combination would produce enough amount of traffic to make the buffer at node m bigger than z .

Now we state the main result of the paper:

Theorem 3.1. (*Buffer asymptotics in networks with fixed routes*) Consider a rate proportional server sharing network with fixed routes and independent $(M/G/\infty)^N$ heterogeneous external inputs with fixed routes specified by with rate parameters r_i , power law tails for the holding times specified by $\beta_i + 1$ at the start node of route i .

Let \mathcal{N}_m denote the flows that use server m and let $r_m^{n,J}$ and $\rho_m^{n,J}$ denote the scaled rates and average loads specified by the fixed points in Lemma 2.1.

Define:

$$R_m^J = \sum_{n \in \mathcal{N}_m} J_n r_m^{n,J} \quad \rho_m^J = \sum_{n \in \mathcal{N}_m} \rho_m^{n,J} \quad \kappa_{J_m} = \sum_{n \in \mathcal{N}_m} J_n \beta_n \quad (14)$$

$$J_0^m = \arg \min_J \{ \kappa_{J_m} : R_m^J > C_m - \rho_m^J \} \quad (15)$$

and assume J_0^m is unique.

Let W_0^m be the stationary workload of node m . Then as $z \rightarrow \infty$,

$$\mathbb{P}\{W_0^m > z\} = O(z^{-\kappa_{J_0^m}})$$

Proof.

Upper bound:

The main idea is to show that if the buffer occupancy at time $t = 0$ for node m reaches a large value of z , then there must have been enough number of long sessions contributing to this. Consider a long session of type n at time t with length $\tau_{t,j}^n > \varepsilon z$ for $j = 1, \dots, \theta_t^n$. Define $S_{t,j}^{n,0} = t$ and if node m_2 is downstream node of node m_1 for type n flow, define $S_{t,j}^{n,m_2} = S_{t,j}^{n,m_1} + W_{S_{t,j}^{n,m_1}}^{m_2}/C_{m_2}$. If m_2 has no predecessor, set $m_1 = 0$. Note that a long session of type n arriving to the network at time t starts being served at node m at time $S_{t,j}^{n,m}$. Consider a fictional queue with infinite buffer and service capacity $\varepsilon_1 > 0$ which serves the long sessions arriving at node m . Let $\bar{S}_{t,j}^{n,m}$ be the time when the service of the considered long session ends at this fictional queue. Also define

$$\xi_n^m = \sum_{t=-1}^{-\infty} \sum_{j=1}^{\theta_t^n} \mathbb{I}_{A_{t,j}^{n,m}}, \quad A_{t,j}^{n,m} = \{S_{t,j}^{n,m} < 0, \bar{S}_{t,j}^{n,m} > 0, \tau_{t,j}^n > \varepsilon z\}$$

Let $\xi^m = (\xi_n^m)$. We claim that

$$\mathbb{P}\{\xi^m = J\} = O(z^{-\kappa_J}) \quad (16)$$

Then we will write

$$\mathbb{P}\{W_0^m > z\} \leq \mathbb{P}\{R_m^{\xi^m} > C_m - \rho_m^{\xi^m}\} + \mathbb{P}\{W_0^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}\}$$

For the first term of right side,

$$\mathbb{P}\{R_m^{\xi^m} > C_m - \rho_m^{\xi^m}\} = \sum_{J \in \mathbb{Z}_+^N} \mathbb{P}\{\xi^m = J, R_m^J > C_m - \rho_m^J\} \leq \mathbb{P}\{\xi^m = J\}$$

From equation (16) and definition of J_0^m , we get

$$\mathbb{P}\{R_m^{\xi^m} > C_m - \rho_m^{\xi^m}\} = O(z^{-\kappa_{J_0^m}})$$

We will then show that

$$\mathbb{P}\{W_0^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}\} = o(z^{-\kappa_{J_0^m}}) \quad (17)$$

and this will complete the proof of the upper bound.

Let us prove claim (16) now. Define $X_t \doteq \sum_{n \in \mathcal{N}} X_t^n$. First, for any $d > 0$, there exists $K > 0$ such that

$$\mathbb{P}\{X(t, t + \tau) > K\tau\} \leq o(\tau^{-d}) \quad (18)$$

as $\tau \rightarrow \infty$. Indeed, consider a queue with service rate $K - 1 > \rho$ with input X_t and let W_t' be its stationary buffer content. Then by using (11), we get

$$\mathbb{P}\{X(t, t + \tau) > K\tau\} = \mathbb{P}\{X(-\tau, 0) > K\tau\} \leq \mathbb{P}\{W_0' > \tau\} = O(\tau^{-\kappa_{J_0}})$$

where $J_0 = \arg \min_J \{\kappa_J : R^J > K - 1 - \rho\}$. Define $|J| = \sum_{n \in \mathcal{N}} J_n$. But $\kappa_{J_0}/\beta_{\min} > |J_0|$ and $|J_0| > \frac{K-1-\rho}{r_{\max}}$. Thus we can choose K such that $\kappa_{J_0} > d$, proving (18).

Now define

$$\tilde{A}_{t,j}^{n,m} = \{S_{t,j}^{n,m} < 0, \bar{S}_{t,j}^{n,m} > 0, \tau_{t,j}^n > \varepsilon z, \bar{S}_{t,j}^{n,m} - S_{t,j}^{n,\pi^n(1)} < (cK)^M(\tau_{t,j}^n)\}$$

where $c > 1/C_k$ for all $k = 1, \dots, M$ and $c > 1/\varepsilon_1$. Here cK will be the lengthening factor due to the scaling that a long session will experience while going through a node, where K is determined by the amount of traffic arriving during its transmission. Also define $\tilde{\xi}_n^m = \sum_{t=-1}^{-\infty} \sum_{j=1}^{\theta_t^n} \mathbb{1}_{\tilde{A}_{t,j}^{n,m}}$. By using (18) and choosing K large enough, we conclude

$$\mathbb{P}\{\xi^m = J\} \sim \mathbb{P}\{\tilde{\xi}^m = J\} \quad (19)$$

thus proving equation (16). Now let $D = (cK)^M + 1$. Define $W_t = (W_t^1, \dots, W_t^M)$. Then,

$$\mathbb{P}\{\tilde{\xi}^m = J\} = \sum_{x \in \mathbb{R}_+^{M|J|}} \sum_{-\mathbf{t} \in Z_+^{|J|}} \mathbb{P}\{W_{t_i} = x_i, c|x_i| + D\tau_{t_i}^{n_i} > -\mathbf{t}_i, \tau_{t_i}^{n_i} > \varepsilon z\}$$

$$i = 1, \dots, |J|, n_i \in \mathcal{N}, \sum_i \mathbb{1}_{\{n_i = n\}} = J_n$$

Above, $x = (x_1, \dots, x_{|J|})$ where $x_i \in \mathbb{R}_+^M$. Now note that W_{t_i} and $\tau_{t_i}^{n_i}$ are independent since W_{t_i} is determined by arrivals before t_i . It is easy to see that

$$\sum_{-t \in Z_+} \mathbb{P}\{x + D\tau_t^n > -t, \tau_t^n > \varepsilon z\} = O(z^{-\beta_n})$$

Furthermore since W_t is stationary, $\mathbb{P}\{W_{t_i} = x_i\} = \mathbb{P}\{W_{\tilde{t}_i} = x_i\}$ where $\tilde{t}_i = t_i - \min_i t_i$. Therefore,

$$\mathbb{P}\{\tilde{\xi}^m = J\} = \sum_{x \in \mathbb{R}_+^{M|J|}} \sum_{-\mathbf{t} \in Z_+^{|J|}} \mathbb{P}\{W_{\tilde{t}_i} = x_i\} O(z^{-\kappa_J}) = O(z^{-\kappa_J})$$

Now we will prove equality (17). Consider the system where all the buffers are empty at time $-T$ and let $W_{T,t} = (W_{T,t}^1, \dots, W_{T,t}^M)$ be the buffer occupancy (workload) at time t for this system. It is known that $W_{T,t} \rightarrow W_t$ a.s. as $T \rightarrow \infty$. Therefore we will first show that equality (17) holds when W_0^m is replaced by $W_{T,0}^m$ and then take the limit $T \rightarrow \infty$.

Let us investigate the total arrival traffic to node m in the time interval $(-T, 0)$ assuming that all the queues are empty at $-T$ and $R_m^{\xi^m} < C_m - \rho_m^{\xi^m}$. Choose $\varepsilon_1 > 0$ such that $2\varepsilon_1 > C_m - \max\{R_m^J + \rho_m^J : R_m^J + \rho_m^J < C_m\}$. If a long session is not in $A_{t,j}^{n,m}$, then it had been served before $t = 0$ in the fictional queue defined above. Consider another queue with capacity $C_m - \varepsilon_1$ which serves all of the

remaining traffic. Note that the buffer content of this queue at $t = 0$ is bigger than $W_{T,0}^m$ provided that $R_m^{\xi^m} < C_m - \rho_m^{\xi^m}$.

Now we assume that the long sessions in $A_{t,j}^{n,m}$ are active in $(-T, 0)$ all the time and the remaining long sessions of other classes are removed. Note that this does not decrease the buffer content at node m . More generally, replacing flows accessing node m by pathwise larger ones and other flows by smaller ones does not decrease the buffer content at node m . This can be seen by a sample path argument. If all the queues are empty at $t = -1$, then from the arguments in Lemma 2.1, the claim will hold at $t = 0$ and an induction on t will complete the proof. Under the above assumption, let $\bar{W}_{T,0}^m$ be the buffer content of the queue with capacity $C_m - \varepsilon_1$. Then,

$$\mathbb{P}\{W_{T,0}^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}\} \leq \mathbb{P}\{\bar{W}_{T,0}^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}\}$$

Define $\mathcal{D}_m = \{R_m^{\xi^m} < C_m - \rho_m^{\xi^m}, \xi^m \text{ sessions are active in } (-T, 0)\}$. Also let $X_m \doteq \sum_{n \in \mathcal{N}_m} X_m^n$ be the total cumulative input to node m . Now assume that

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} X_k^n(-T, -t) - (r_k^{n, \xi^m} + \rho_k^{n, \xi^m} + \varepsilon)(T - t) > \varepsilon_2 z, \mathcal{D}_m\right\} \preceq z^{-d} \quad (20)$$

and

$$\mathbb{P}\left\{\inf_{0 \leq t \leq T} X_k^n(-T, -t) - (r_k^{n, \xi^m} + \rho_k^{n, \xi^m} - \varepsilon)(T - t) < -\varepsilon_2 z, \mathcal{D}_m\right\} \preceq z^{-d} \quad (21)$$

for any $d > 0$. Note that this is true for $k = \pi_1^n$. Indeed, if \mathcal{D}_m holds, then the long processes are constant ($= r_k^{n, \xi^m}$) in $(-T, 0)$ for type n flow. Furthermore, from Lemmas 2.2 and 2.3, a short process $X^{n,L}$ differs from the constant processes $\rho_n - \delta$ and $\rho_n + \delta$ with probability $o(z^{-\bar{d}})$ for any $\delta > 0$. Here \bar{d} can be chosen arbitrarily big. Then,

$$\begin{aligned} & \mathbb{P}\left\{\sup_{0 \leq t_1 \leq T} Y_k^n(-T, t) - \left(\frac{r_m^{n, \xi^m} + \rho_m^{n, \xi^m}}{\max(C_k, R_m^{\xi^m} + \rho_m^{\xi^m})} + \varepsilon_1\right)(T - t) > \varepsilon_2 z, \mathcal{D}_m\right\} \\ & \leq \mathbb{P}\left\{\sup_{0 \leq t \leq T} X_k^n(-T, -t) - (r_k^{n, \xi^m} + \rho_k^{n, \xi^m} + \varepsilon_3)(T - t) > \varepsilon_4 z, \mathcal{D}_m\right\} + \\ & \quad \sum_{j \in \mathcal{N}_k, j \neq n} \mathbb{P}\left\{\inf_{0 \leq t \leq T} X_k^j(-T, -t) - (r_k^{j, \xi^m} + \rho_k^{j, \xi^m} - \varepsilon_3)(T - t) - \varepsilon_4 z, \mathcal{D}_m\right\} \\ & \leq o(z^{-d}) \end{aligned}$$

We will choose $\varepsilon_3 = \varepsilon_1 / \text{card}(\mathcal{N}_k)$ and $\varepsilon_4 = (\varepsilon_2 - M\delta_1) / \text{card}(\mathcal{N}_k)$. Above we used the fact that the output of node k in the interval $(-T, -t]$ is equal to the arrival in an interval $(-T, t']$. Note that the time being discrete is not a problem here since there is an $\varepsilon_2 z$ term and z is taken to be large. The same argument can be used to show that

$$\begin{aligned} & \mathbb{P}\left\{\inf_{0 \leq t \leq T} Y_k^n(-T, -t) - \left(\frac{r_m^{n, \xi^m} + \rho_m^{n, \xi^m}}{\max(C_k, R_m^{\xi^m} + \rho_m^{\xi^m})} - \varepsilon_1\right)(T - t) < -\varepsilon_2 z, \mathcal{D}_m\right\} \\ & \leq o(z^{-d}) \end{aligned}$$

From Lemma 2.1, the equations (20) and (21) should hold for all n, k . This can be more formally shown by considering the values

$$\bar{\alpha}(n, k) \doteq$$

$$\lim_{z \rightarrow \infty} \frac{1}{\log z} \log \left\{ \mathbb{P}\left\{\sup_{0 \leq t \leq T} X_k^n(-T, -t) - (r_k^{n, \xi^m} + \rho_k^{n, \xi^m} + \varepsilon_1)(T - t) > \varepsilon_2 z, \mathcal{D}_m\right\} \right\}$$

and defining similarly $\underline{\alpha}(n, k)$ where sup is replaced with inf. These expressions evaluate to ∞ when $k = \pi_1^n$. Above arguments show that it is also true for all n, k at the outputs assuming it for the inputs. Thus $\bar{\alpha}(n, k) = \underline{\alpha}(n, k) = \infty$ is a fixed point solution of these relations as given in Lemma 2.1. Since there can only be one solution, we conclude that the equation (20) is valid for all n, k . Thus for the input to node m , we have

$$\begin{aligned} & \mathbb{P}\{\sup_{0 < t \leq T} X_m(-t, 0) - (R_m^{\xi^m} + \rho_m^{\xi^m} + \varepsilon_1)t > \varepsilon_2 z, \mathcal{D}_m\} \\ & \leq \mathbb{P}\{X_m(-T, 0) - (R_m^{\xi^m} + \rho_m^{\xi^m} + \varepsilon_1)T > 0.5\varepsilon_2 z, \mathcal{D}_m\} \\ & \quad + \mathbb{P}\{\inf_{0 < t \leq T} X_m(-T, t) - (R_m^{\xi^m} + \rho_m^{\xi^m} + \varepsilon_1)(T - t) < -0.5\varepsilon_2 z, \mathcal{D}_m\} \\ & \preceq o(z^{-d}) \end{aligned}$$

for any $d > 0$. Remember that ρ_m^{n, J_0^m} and r_m^{n, J_0^m} are continuous functions of ρ_k, r_k 's. Then,

$$\begin{aligned} & \mathbb{P}\{\bar{W}_{T,0}^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}, T_0 < T\} \\ & \leq \mathbb{P}\{\sup_{0 < t \leq T} X_m(-t, 0) - (R_m^{\xi^m} + \rho_m^{\xi^m} + \varepsilon_1)t > \varepsilon_2 z, \mathcal{D}_m\} \preceq o(z^{-d}) \end{aligned}$$

and finally by taking $T \rightarrow \infty$, we get

$$\mathbb{P}\{W_0^m > z, R_m^{\xi^m} < C_m - \rho_m^{\xi^m}\} \leq o(z^{-\kappa J_0^m})$$

and this proves the claim in equation (17).

Lower bound:

Let $A^J(x, \alpha)$ be the event that exactly J long sessions start in $(-x, -(1 - \alpha)x]$ and are still active at $t = 0$ where $0 < \alpha < 1$. First note that, for every class n , the number of such sessions is a Poisson r.v. with parameter $\sum_{t=-x+1}^{-(1-\alpha)x} \lambda_n \mathbb{P}\{\tau_n > -t\}$ which is $O(x^{-\beta_n})$. From the independence of classes, it is easy to see that $\mathbb{P}\{A^J(x, \alpha)\} \sim O(x^{-\kappa J})$. We will now show that

$$\mathbb{P}\{A^{J_0^m}(x, \alpha)\} \preceq \mathbb{P}\{W_0^m > z\}$$

for some $\alpha > 0$ and $x = bz$ with $b > 0$. First, from the upper bound proof,

$$\mathbb{P}\{W_{-x}^k > \delta_1 z\} \preceq O(z^{-\gamma}) \tag{22}$$

for some $\gamma > 0$ and any $k = 1, \dots, M$. Note that $\{W_{-x}^k > \delta_1 z\}$ and $A^{J_0^m}(x, \alpha)$ are independent because W_{-x}^k is determined by sessions which arrived before $-x + 1$. Now define the following two events:

$$\begin{aligned} B_1 &= \{\text{there are active long sessions at time } t = -x\} \\ B_2 &= \{\text{a long session other than } J_0^m \text{ arrives between } t = -x \text{ and } t = 0\} \end{aligned}$$

The arrival process of long sessions is also Poisson with rate $\lambda_m \mathbb{P}\{\tau_m > \varepsilon z\}$. Since the arrivals at different times are independent, it is easy to see that

$$\mathbb{P}\{B_1 | A^{J_0^m}(x, \alpha)\} \leq O(z^{-\beta_{\min}}) \quad \text{and} \quad \mathbb{P}\{B_2 | A^{J_0^m}(x, \alpha)\} \leq O(xz^{-\beta_{\min}-1}).$$

These and equation (22) give

$$\mathbb{P}\{A^{J_0^m}(x, \alpha)\} \sim \mathbb{P}\{A^{J_0^m}(x, \alpha), \bar{B}_1, \bar{B}_2, W_{-x}^k < \delta_1 z, \forall k = 1, \dots, M\} \tag{23}$$

where \bar{B}_1, \bar{B}_2 are complements of B_1, B_2 . In other words, we can assume that all the buffers are almost empty at $t = -x$ regarding their contribution to a buffer level exceeding z at $t = 0$ and the only active long sessions during $(-x, 0)$ are the ones of J_0^m . Let

$$\mathcal{A}_m = \{A^{J_0^m}(x, \alpha), \bar{B}_1, \bar{B}_2, W_{-x}^k < \delta_1 z, \forall k = 1, \dots, M\}$$

Let $T = (1 - \alpha)x$. Now assume that

$$\mathbb{P}\left\{\sup_{0 \leq t_1 \leq t_2 \leq T} X_k^n(-t_2, -t_1) - (r_k^{n, J_0^m} + \rho_k^{n, J_0^m} + \varepsilon)(t_2 - t_1) > \varepsilon_2 z, \mathcal{A}_m\right\} \preceq z^{-d}$$

and

$$\mathbb{P}\left\{\inf_{0 \leq t_1 \leq t_2 \leq T} X_k^n(-t_2, -t_1) - (r_k^{n, J_0^m} + \rho_k^{n, J_0^m} - \varepsilon)(t_2 - t_1) < -\varepsilon_2 z, \mathcal{A}_m\right\} \preceq z^{-d}$$

for any $d > 0, \varepsilon, \varepsilon_2 > 0$. Note that this is true when $k = \pi_1^n$. Indeed, if \mathcal{A}_m holds, then the long processes are constant ($= r_k^{n, J_0^m}$) in $(-T, 0)$ for type n flow. Furthermore, from Lemmas 2.2 and 2.3, a short process $X^{n, L}$ differs from the constant processes $\rho_n - \delta$ and $\rho_n + \delta$ with probability $o(z^{-\bar{d}})$ for any $\delta > 0$. Here \bar{d} can be chosen arbitrarily big and since $T = (1 - \alpha)bz$, we can take $d < \bar{d} - 1$. Then,

$$\begin{aligned} & \mathbb{P}\left\{\sup_{0 \leq t_1 \leq t_2 \leq T} Y_k^n(-t_2, -t_1) - \left(\frac{r_m^{n, J_0^m} + \rho_m^{n, J_0^m}}{\max(C_k, R_m^{J_0^m} + \rho_m^{J_0^m})} + \varepsilon_1\right)(t_2 - t_1) > \varepsilon_2 z, \mathcal{A}_m\right\} \\ & \leq \mathbb{P}\left\{\sup_{0 \leq t_1 \leq t_2 \leq T} X_k^n(-t_2, -t_1) - (r_k^{n, J_0^m} + \rho_k^{n, J_0^m} + \varepsilon_3)(t_2 - t_1) > \varepsilon_4 z, \mathcal{A}_m\right\} + \\ & \sum_{j \in \mathcal{N}_k, j \neq n} \mathbb{P}\left\{\inf_{0 \leq t_1 \leq t_2 \leq T} X_k^j(-t_2, -t_1) - (r_k^{j, J_0^m} + \rho_k^{j, J_0^m} - \varepsilon_3)(t_2 - t_1) - \varepsilon_4 z, \mathcal{A}_m\right\} \\ & \leq o(z^{-d}) \end{aligned}$$

We will choose $\varepsilon_3 = \varepsilon_1 / \text{card}(\mathcal{N}_k)$ and $\varepsilon_4 = (\varepsilon_2 - M\delta_1) / \text{card}(\mathcal{N}_k)$. Above, we used the fact that the output of node k in the interval $(-t_2, -t_1]$ is equal to the arrival in an interval $(-t'_2, t'_1]$. Note that the error induced here due to the discreteness of time is not a problem since there is an $\varepsilon_2 z$ term and z is taken to be large. The same argument can be used to show that

$$\begin{aligned} & \mathbb{P}\left\{\inf_{0 \leq t_1 \leq t_2 \leq T} Y_k^n(-t_2, -t_1) - \left(\frac{r_m^{n, J_0^m} + \rho_m^{n, J_0^m}}{\max(C_k + R_m^{J_0^m} + \rho_m^{J_0^m})} - \varepsilon_1\right)(t_2 - t_1) < -\varepsilon_2 z, \mathcal{A}_m\right\} \\ & \leq o(z^{-d}) \end{aligned}$$

Then by using Lemma 2.1 as was done in the upper bound part, we conclude

$$\begin{aligned} & \mathbb{P}\{X_m(-T, 0) - (R_m^{J_0^m} + \rho_m^{J_0^m} - \varepsilon_1)T < -\varepsilon_2 z - \alpha T(R^J + \rho^J + 1), \mathcal{A}_m\} \\ & \leq o(z^{-d}) \end{aligned}$$

for any $d > 0$. Then

$$\begin{aligned} & \mathbb{P}\{W_0^m > z\} \geq \mathbb{P}\{X_m(-x, 0) > C_m x + z\} \geq \mathbb{P}\{X_m(-x, 0) > C_m x + z, \mathcal{A}_m\} \\ & \geq \mathbb{P}\{(R_m^{J_0^m} + \rho_m^{J_0^m} - \varepsilon_1 - \delta_3)T > C_m x + (1 + \varepsilon_2)z, \mathcal{A}_m\} + o(z^{-d}) \\ & \geq \mathbb{P}\{A^{J_0^m}(x, \alpha)\} \end{aligned}$$

For the last inequality, we used $R_m^{J_0^m} + \rho_m^{J_0^m} > C_m$ and chose $\alpha, \varepsilon_1, \varepsilon_2$ small enough and b large enough so that

$$(R_m^{J_0^m} + \rho_m^{J_0^m})(1 - \alpha) > C_m + \frac{1 - \varepsilon_2}{b}$$

Combining above with equation (23) gives

$$\mathbb{P}\{W_0^m > z\} \succeq \mathbb{P}\{A^{J_0^m}(x, \alpha)\} \sim O(z^{-\kappa_{J_0^m}})$$

□

Remark 3.1. *The uniqueness of J_0^m was assumed for convenience and is not necessary. It can easily be seen that the results and proofs still hold when there are many optimal configurations of long sessions.*

Remark 3.2. *If there are input flows to the network with light tailed session lengths such that $\lim_{x \rightarrow \infty} \log \mathbb{P}\{\tau > x\} / \log x = -\infty$, then they can be ignored in calculating the buffer asymptotics. Such an input can be replaced with a Pareto tailed version with arbitrarily large value β without decreasing the probability of large buffer occupancies. Therefore, in determining J_0^m , none of these inputs need to be considered.*

Remark 3.3. *If the session lengths are regularly varying (i.e. $\lim_{x \rightarrow \infty} \mathbb{P}\{\tau_n > tx\} / \mathbb{P}\{\tau_n > x\} = t^{-(1+\beta_n)}$ for $\beta_n > 0$), it can be shown that the buffer occupancy distributions are also regularly varying with the parameters as found in the main result.*

We can also find the joint distribution of buffer asymptotics. We only state the result since the proof follows mutatis mutandis as above.

Corollary 3.1. *Let $\mathcal{S} \subset \{1, \dots, M\}$ and $a_m > 0, m \in \mathcal{S}$. Then as $z \rightarrow \infty$,*

$$\mathbb{P}\{W_0^m > a_m z, m \in \mathcal{S}\} \sim O(z^{-\kappa_{\mathcal{S}}})$$

where $\kappa_{\mathcal{S}} = \max\{\kappa_{J_0^m} \mid m \in \mathcal{S}\}$

4 Examples

In order to illustrate the results, in particular how the rates $r_m^{n,J}, \rho_m^{n,J}$ are determined, we consider two examples of a simple network with two nodes. The first example is a feedforward network while the second example is a network where individual routes have no loops but the network is not feedforward.

Example 1: There are three classes of traffic, one of which uses resources from both nodes. The schema is illustrated in the figure below.

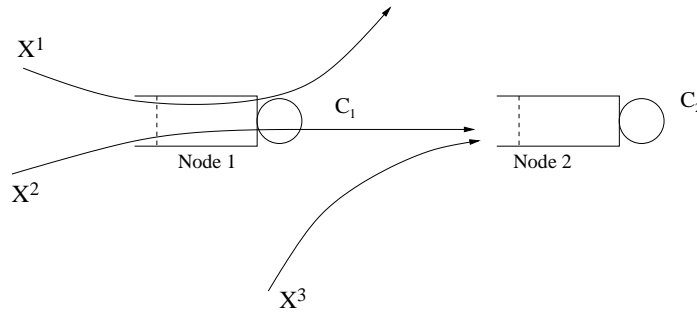


Figure 3: Two node feedforward network

Now we calculate the large buffer asymptotics at the second node. First,

$$\begin{aligned} r_2^{1,J} &= 0, & r_2^{2,J} &= \frac{r_2 C_1}{\max(J_1 r_1 + J_2 r_2 + \rho_1 + \rho_2, C_1)}, & r_2^{3,J} &= r_3 \\ \rho_2^{1,J} &= 0, & \rho_2^{2,J} &= \frac{\rho_2 C_1}{\max(J_1 r_1 + J_2 r_2 + \rho_1 + \rho_2, C_1)}, & \rho_2^{3,J} &= \rho_3 \end{aligned}$$

and thus

$$J_0^2 = \arg \min_{J \in \mathbb{Z}_+^3} \left\{ \sum_{m=1}^3 J_m \beta_m : \frac{(J_2 r_2 + \rho_2) C_1}{\max(C_1, J_1 r_1 + J_2 r_2 + \rho_1 + \rho_2)} + J_3 r_3 + \rho_3 > C_2 \right\}$$

In special cases, J_0^2 can be obtained easily. For example, if classes 2 and 3 have the same transmission and decay rates ($r_2 = r_3, \beta_2 = \beta_3$), then one optimal configuration is $J_0^2 = (0, 0, \lceil \frac{C_2 - \rho_2 - \rho_3}{r_3} \rceil)$. This is because class 1 sessions do not contribute to the buffer occupancy at node 2 and we can take all the long sessions to be of class 3.

Example 2: In this example, we consider a non-feedforward network with two nodes and two types of traffic as illustrated in Fig. 4.

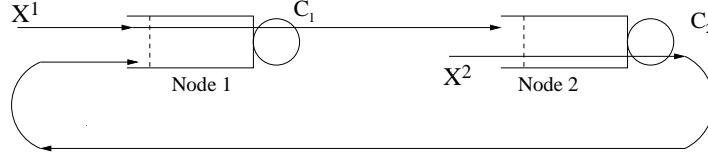


Figure 4: Two node non-feedforward network

In this case, we obtain the following:

$$\begin{aligned} r_2^{1,J} &= r_1 / \max(J_1 r_1 + J_2 r_1^{2,J} + \rho_1 + \rho_1^{2,J}, C_1), & r_2^{2,J} &= r_2, \\ r_1^{2,J} &= r_2 / \max(J_1 r_2^{1,J} + J_2 r_2 + \rho_2 + \rho_2^{1,J}, C_2), & r_1^{1,J} &= r_1, \\ \rho_2^{1,J} &= \rho_1 / \max(J_1 r_1 + J_2 r_1^{2,J} + \rho_1 + \rho_1^{2,J}, C_1), & \rho_2^{2,J} &= \rho_2, \\ \rho_1^{2,J} &= \rho_2 / \max(J_1 r_2^{1,J} + J_2 r_2 + \rho_2 + \rho_2^{1,J}, C_2), & \rho_1^{1,J} &= \rho_1, \end{aligned}$$

Hence

$$\kappa_{J_0^2} = \min \left\{ \sum_{m=1}^3 J_m \beta_m : J \in \Omega_1 \cup \Omega_2 \right\} \quad (24)$$

where

$$\Omega_1 = \left\{ J \in \mathbb{Z}_+^3 : J_1 r_1 + \rho_1 + \frac{(J_2 r_2 + \rho_2) C_2}{J_1 r_1 + \rho_1 + J_2 r_2 + \rho_2} < C_1, J_1 r_1 + \rho_1 + J_2 r_2 + \rho_2 > C_2 \right\}$$

$$\Omega_2 = \left\{ J \in \mathbb{Z}_+^3 : J_1 r_1 + \rho_1 + \beta(J_2 r_2 + \rho_2) > C_1, \alpha(J_1 r_1 + \rho_1) + J_2 r_2 + \rho_2 > C_2 \right\}$$

Here $\alpha = C_1 / (J_1 r_1 + \rho_1 + \beta(J_2 r_2 + \rho_2))$ and $\beta = C_2 / (\alpha(J_1 r_1 + \rho_1) + J_2 r_2 + \rho_2)$. Note that Ω_1 corresponds to the situation where large buffer occupancy happens only at the second node and Ω_2 is where it happens at both nodes. In finding the optimal point in equation (24), we need to check whether a combination (J_1, J_2) is inside Ω_1 or Ω_2 . For Ω_1 , this is easy. For Ω_2 , first α and β are algebraically or numerically solved and then substituted to the constraint in the definition of Ω_2 . If classes 1 and 2 have the same transmission and decay rates ($r_1 = r_2, \beta_1 = \beta_2$), then it can be seen that it is enough to carry out the above calculations assuming only the long sessions of class 2. In addition, if $C_1 > C_2$, then $J_0^2 = (0, \lceil \frac{C_2 - \rho_1 - \rho_2}{r_2} \rceil)$ since now the most likely large buffer occupancy at the second node will happen without one at the first node.

5 Conclusion

In this paper we considered a loop-free networks of rate proportional servers in discrete-time with heterogeneous $(M/G/\infty)^N$ inputs with long-tailed Pareto holding times. We have shown that the

buffer occupancy distribution is also Pareto-like and determined its parameters. The continuous time case can be readily treated with slight changes in the technical details of the proofs. On the other hand, it seems difficult to obtain finer asymptotics, i.e., determining the prefactor constants because we cannot parametrically model the traffic inside the network.

A further extension of the model to cope with more general models will require different tools since our proofs depended very heavily on the compound Poisson structure of the input processes. Nevertheless the Poisson arrival of flows or sessions is by well motivated by the Poisson superposition theorem for stationary independent point processes.

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A Proof of Lemma 2.1

We will show that $r_m^{n,J}$ and $\rho_m^{n,J}$ form a unique fixed point of a function for a given set of values of r_n, ρ_n . Let us now describe this function: Let $q = 2 \sum_{n=1}^N l_n$, $s_1 = 1$, $s_n = 2 \sum_{k=1}^{n-1} l_k$, $1 < n \leq N$ and define

$$\Omega = \{v \in \mathbb{R}^q : v_{s_n+j} \in [0, C_{\pi_j^n}], n = 1, \dots, N, j = 1, \dots, l_n - 1\}$$

Here vector v corresponds to the transmission rates and average loads of the flows at all nodes. Now for $J \in \mathbb{Z}_+^N$, define the function $T^J : \Omega \rightarrow \Omega$ such that

$$T^J(v)_{s_n+j+1} = \frac{v_{s_n+j}}{\max(\sum_{k \in \mathcal{N}_{\pi_j^n}} J_k v_{s_k+i(k,j,n)} + v_{s_k+l_k+i(k,j,n)}, C_{\pi_j^n})}, \quad (25)$$

$$j = 0, \dots, l_n - 2, l_n, l_n + 1, \dots, 2l_n - 2$$

where $i(k, j, n)$ is chosen such that $\pi_i^k = \pi_j^n$ with $\bar{j} = j \bmod(l_n)$.

The function T^J expresses the input rates to a node in terms of the output rates of the upstream nodes. Since each input is either an external flow or the output of another node, we must have the relation $T^J(v) = v$. Now we need to show that this is indeed the case and for a fixed $2N$ dimensional vector w given by $w_n = v_{s_n}$ and $w_{n+N} = v_{s_n+l_n}$, i.e., for fixed values of the external input rates, this solution is unique. It is easy to check that $T^J(\Omega) \subset \Omega$. Let T_w^J be equal to T^J for a fixed w , i.e., $T_w^J = T^J|_{\Omega_w}$ with $\Omega_w = \{v \in \Omega : v_{s_n} = w_n, v_{s_n+l_n} = w_{n+N}\}$. Now we will show that T_w^J has a unique fixed point. From an extension of Banach fixed point theorem [10, p.187], it is enough to show that T_w^J is a condensing mapping which means that for a given metric d and for $u, v \in \Omega_w$, $d(T_w^J(u), T_w^J(v)) < d(u, v)$. We choose the metric d to be the one corresponding to the L_2 norm. To prove that T_w^J is condensing, it is sufficient to show that the transformation at each node between input and output rates is a condensing mapping. Indeed T_w^J can be written as a disjoint sum of such transformations by choosing the appropriate permutation of $\{v_i\}$. Hence we will consider a generic mapping of the form

$$F : D \rightarrow D, F(v)_j = \frac{v_j C}{\max(\sum_{i=1}^p J_i v_i, C)}$$

where $D \subset \mathbb{R}_+^p$ is a compact set for some $p > 0$ and $J \in \mathbb{Z}_+^p$. The compactness condition on the domain D is justified because a flow other than an external input is output of a node and hence it has a bounded rate. Define $\|v\| = \sum_{i=1}^p J_i v_i$. Take $u, v \in D$. If $\|u\| < C, \|v\| < C$, then $F(u) - F(v) = u - v$. If $\|u\| < C, \|v\| > C$, take $z = tu + (1-t)v$, $t \in (0, 1)$ such that $\|z\| = C$. Then,

$$F(u) - F(v) = F(u) - F(z) + F(z) - F(v) = u - z + F(z) - F(v)$$

Now consider the case $\|u\| > C, \|v\| > C$. Then \mathcal{J}_F , the Jacobian of F , is given by

$$\mathcal{J}_F(v) = \frac{C}{\|v\|^2} A \text{ where } A_{ii} = \|v\| - J_i v_i, A_{ij} = -J_j v_i, i \neq j$$

Thus $A = \|v\| I_p - B$ where $B_{ij} = J_j v_i$ and I_p is the $p \times p$ identity matrix. Note that the matrix B has rank "1" and thus it has one eigenvalue at its trace equal to $\|v\|$ with the remaining eigenvalues being 0. By taking an invertible matrix G such that $G^{-1}BG$ is Jordan form [16] of B , it follows that $G^{-1}AG$ has one eigenvalue at 0 and $p-1$ eigenvalues at $\|v\|$. Therefore the largest eigenvalue of $\mathcal{J}_F(v)$ is $C/\|v\|$ which is less than 1. This implies that F is a condensing mapping.

As mentioned above, T_w^J can be written as the disjoint sum of F type transformations. Therefore T_w^J is also a condensing mapping and has a unique fixed point $v^0(w)$ provided that $v^0(w)$ satisfies $\sum_{n \in \mathcal{M}_k} J_n v^0(w)_{s_n+n_k} + v^0(w)_{s_n+l_n n_k} > C_k$ for at least one node k . Here n_k is chosen such that $\pi_{n_k}^n = k$. If this does not hold, then a fixed point must satisfy $v^0(w)_j = w_n$ for $s_n \leq j < s_n + l_n$ and $v^0(w)_j = w_{n+N}$ for $s_n + l_n \leq j < s_{n+1}$. Therefore if $\sum_{n \in \mathcal{M}_k} J_n v^0(w)_{s_n+n_k} + v^0(w)_{s_n+l_n n_k} < C_k$ holds for every node k , then $v^0(w)$ is a unique fixed point. Thus we have shown that for every w , there exists a unique $v^0(w)$ satisfying $T_w(v^0(w)) = v^0(w)$. In other words, for a given set of transmission rates r_n and average loads ρ_n of the external inputs, the corresponding internal ones are uniquely defined as in equations (4) and (5). Furthermore $v^0(w)$ is also a continuous function of w . Indeed assume that this is not true and there exists $n \rightarrow \infty$, $w_n \rightarrow w$ but $v^0(w_n) \not\rightarrow v^0(w)$. But since $v^0(w_n)$ lies in a compact region, there exists \bar{v} s.t. $v^0(w_{n_k}) \rightarrow \bar{v} \in \Omega_w$ and $\bar{v} \in \Omega_w$. Since T is continuous, $T(\bar{v}) = \bar{v}$. But this is in contradiction to the uniqueness of the fixed point in Ω_w and thus proves the continuity of $v^0(\cdot)$. \square